Jawaban UTS Matematika Informatika 3

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1. By looking at the first few sums, we guess that the sum is n/(n + 1). We prove this by induction. It is clear for n = 1, since there is just one term, 1/2. Suppose that

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

   We want to show that

$$\left[\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

   Starting from the left, we replace the quantity in brackets by k/(k+1) (by the inductive hypothesis), and then do the algebra

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

   yielding the desired expression.

2. The statement is true for the base case, n = 0, since 3 | 0. Suppose that 3 | (k^3 + 2k). We must show that 3 | (($(k + 1)^3$ + 2(k + 1)) . If we expand the expression in question, we obtain k^3 + $3k^2$ + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1). By the inductive hypothesis, 3 divides k^3 + 2k, and certainly 3 divides 3(k^2 + k + 1), so 3 divides their sum, and we are done.

3. a) The product rule applies, since the student will perform each of 10 tasks, one after the other. There are 4 ways to do each task. Therefore there are 4 • 4 • • • 4 = 4^{10} = 1,048,576 ways to answer the questions on the test.

   b) This is identical to part (a), except that now there are 5 ways to answer each question - give any of the 4 answers or give no answer at all. Therefore there are 5^{10} = 9,765,625 ways to answer the questions on the test.

4. By the product rule there are 12 • 2 • 3 = 72 different types of shirt.

5. a) There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains two pigeons. By the pigeonhole principle the answer is 3. If three socks are taken from the drawer, at least two must have the same color. On the other hand two socks are not enough, because one might be brown

and the other black. Note that the number of socks was irrelevant (assuming that it was at least 3).

b) He needs to take out 14 socks in order to insure at least two black socks. If he does so, then at most 12 of them are brown, so at least two are black. On the other hand, if he removes 13 or fewer socks, then 12 of them could be brown, and he might not get his pair of black socks. This time the number of socks did matter.

6. We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 16, namely {1, 15}, {3, 13}, {5, 11}, and {7, 9}. If we select five numbers from the set {1, 3, 5, 7, 9, 11, 13, 15}, then at least two of them must fall within the same subset, since there are only four subsets. Two numbers in the same subset are the desired pair that add up to 16. We also need to point out that choosing four numbers is not enough, since we could choose {1,3,5,7}, and no pair of them add up to more than 12.

7. We simply plug into the formula $P(n,r) = n(n - 1)(n - 2) \cdots (n - r + 1)$, given in Theorem 1. Note that there are r terms in this product, starting with n. This is the same as $P(n,r) = n!/(n - r)!$, but the latter formula is not as nice for computation, since it ignores the fact that each of the factors in the denominator cancels one factor in the numerator. Thus to compute n! and (n - r)! and then to divide is to do a lot of extra arithmetic. Of course if the denominator is 1, then there is no extra work, so we note that $P(n, n) = P(n, n - 1) = n!$.

a) $P(6,3) = 6 \cdot 5 \cdot 4 = 120$ b) $P(6,5) = 6! = 720$ c) $P(8,l) = 8$

d) $P(8,5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ e) $P(8,8) = 8! = 40,320$ f) $P(10,9) = 10! = 3,628,800$

8. a) To specify a bit string of length 12 that contains exactly three 1's, we simply need to choose the three positions that contain the 1's. There are $C(12,3) = 220$ ways to do that.

b) To contain at most three 1's means to contain three 1's, two 1's, one 1, or no 1's. Reasoning as in part (a), we see that there are $C(12,3) + C(12,2) + C(12,1) + C(12,0) = 220+66+12 +1 = 299$ such strings.

c) To contain at least three 1's means to contain three 1's, four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, 10 1's, 11 1's, or 12 1's. We could reason as in part (b), but we would have too many numbers to add. A simpler approach would be to figure out the number of ways not to have at least three 1's (i.e., to have two 1's, one 1, or no 1's) and then subtract that from 2^{12}, the total number of bit strings of length 12. This way we get $4096 - (66 + 12 + 1) = 4017$.

d) To have an equal number of 0's and 1's in this case means to have six 1's. Therefore the answer is $C(12,6) = 924$.

9. a) If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items - the letters A, E, F, and G, and the superletter BCD. Therefore the answer is $P(5,5) = 5! = 120$.

b) Reasoning as in part (a), we see that the answer is P(4,4) = 4! = 24.

c) As in part (a), we glue BA into one item and glue GF into one item. Therefore we need to permute five items, and there are P(5,5) = 5! = 120 ways to do it.

d) This is similar to part (c). Glue ABC into one item and glue DE into one item, producing four items, so the answer is P(4,4) = 4! = 24.

e) If both ABC and CDE are substrings, then ABCDE has to be a substring. So we are really just permuting three items: ABCDE, F, and G. Therefore the answer is P(3,3) = 3! = 6.

f) There are no permutations with both of these substrings, since B cannot be followed by both A and E at the same time.

10. There are 6 choices each of 7 times, so the answer is 6^7 = 279,936.

11. We assume that the jobs and the employees are distinguishable. For each job, we have to decide which employee gets that job. Thus there are 5 ways in which the first job can be assigned, 5 ways in which the second job can be assigned, and 5 ways in which the third job can be assigned. Therefore, by the multiplication principle (just as in Exercise 1) there are 5^3 = 125 ways in which the assignments can be made. (Note that we do not require that every employee get at least one job.)

12. Let b_1, b_2, ..., b_8 be the number of bagels of the 8 types listed (in the order listed) that are selected. Order does not matter: we are presumably putting the bagels into a bag to take home, and the order in which we put them there is irrelevant.

    a) If we want to choose 6 bagels, then we are asking for the number of nonnegative solutions to the equation $b_1 + b_2 + \cdots + b_8 = 6$. Theorem 2 applies, with n = 8 and r = 6, giving us the answer C(8 + 6 - 1,6) = C(13,6) = 1716.

    b) This is the same as part (a), except that r = 12 rather than 6. Thus there are C(8 + 12 - 1,12) = C(19,12) = C(19,7) = 50,388 ways to make the selection. (Note that C(19,7) was easier to compute than C(19,12), and since they are equal, we chose the latter form.)

    c) This is the same as part (a), except that r = 24 rather than 6. Thus there are C(8 + 24 - 1,24) = C(31,24) = C(31,7) = 2,629,575 ways to make the selection.

    d) This one is more complicated. Here we want to solve the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the constraint that each $b_i \geq 1$. We reduce this problem to the form in which Theorem 2 is applicable with the following trick. Let $b'_1 = b_i - 1$; then b_i represents the number of bagels of type i, in excess of the required 1, that are selected. If we substitute $b_i = b'_1 + 1$ into the original equation, we obtain $(b'_1 + 1) + (b'_2 + 1) + \cdots + (b'_8 + 1) = 12$, which reduces to $b'_1 + b'_2 \cdots + b'_8 =$ 4. In other words, we are asking how many ways are there to choose the 4 extra bagels (in excess of the required 1 of each type) from among the 8 types, repetitions allowed. By Theorem 2 the number of solutions is C(8 + 4 - 1, 4) = C(11, 4) = 330.

e) This final part is even trickier. First let us ignore the restriction that there can be no more than 2 salty bagels (i.e., that $b_4 \le 2$). We will take into account, however, the restriction that there must be at least 3 egg bagels (i.e., that $b_3 \ge 3$). Thus we want to count the number of solutions to the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the condition that $b_i \ge 0$ for all i and $b_3 \ge 3$. As in part (d), we use the trick of choosing the 3 egg bagels at the outset, leaving only 9 bagels free to be chosen; equivalently, we set $b'_3 = b_3 - 3$, to represent the extra egg bagels, above the required 3, that are chosen. Now Theorem 2 applies to the number of solutions of $b_1 + b_2 + b'_3 + b_4 + \cdots + b_8 = 9$, so there are C(8 + 9 - 1,9) = C(16,9) = C(16,7) = 11,440 ways to make this selection.
Next we need to worry about the restriction that $b_4 \le 2$. We will impose this restriction by subtracting from our answer so far the number of ways to violate this restriction (while still obeying the restriction that $b_3 \ge 3$). The difference will be the desired answer. To violate the restriction means to have $b_4 \ge 3$. Thus we want to count the number of solutions to $b_1 + b_2 + \cdots + b_8 = 12$, with $b_3 \ge 3$ and $b_4 \ge 3$. Using the same technique as we have just used, this is equal to the number of nonnegative solutions to the equation $b_1 + b_2 + b'_3 + b'_4 + b_5 + \cdots + b_8 = 6$ (the 6 on the right being 12-3-3). By Theorem 2 there are C(8 + 6 - 1, 6) = C(13,6) = 1716 ways to make this selection. Therefore our final answer is 11440 - 1716 = 9724.